Let us have the signal s(t). Let us apply the discrete Fourier transformation over it. The discrete $s_d(t)$ signal, which is in respect to the analog signal s(t) is:

$$s_d(t) = \sum_{n=0}^{\infty} s(nT).\delta(t-nT) \qquad \dots (1)$$

where:

$$\delta(t-nT) = \begin{cases} 1, t = nT \\ 0, t \neq nT. \end{cases} \qquad \dots (2)$$

We apply the discrete Fourier transformation over (1):

$$\dot{S}(\omega) = \int_0^{\infty} s_d(t).e^{-j\omega t}dt = \int_0^{\infty}\sum_{n=0}^{\infty} s(nT).\delta(t-nT).e^{-j\omega t}dt = \sum_{n=0}^{\infty} s(nT)\int_0^{\infty} e^{-j\omega t}.\delta(t-nT)dt \qquad \dots (3)$$

As it is well known that:

$$\int_0^{\infty} e^{-j\omega t}.\delta(t-nT)dt = e^{-j\omega nT} \qquad \dots (4)$$

Then

$$\dot{S}(\omega) = \sum_{n=0}^{\infty} s(nT).e^{-j\omega nT} \qquad \dots (5)$$

The following frequency is: $\omega = k\Omega = \frac{2\pi k}{NT}$ … (6)

$$\dot{S}(k\Omega) = \sum_{n=0}^{N-1} s(nT).e^{-j\frac{2\pi}{N}kn}; (k = 0, \pm 1, \pm 2, ..., \pm N/2) \qquad \dots (7)$$

That is trivial discrete Fourier transformation in forward direction.
From the other hand if $s(nT) = \sin(\Omega_0 n)$, where $\Omega_0 = 2\pi m/N$, and *m* and *N* are integer numbers, we can make expansion over that signal by Fourier transformation or by easier way using the formula $\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$, (comes from Euler's formula $e^{jx} = \cos(x) + j\sin(x)$):

$$s(nT) = \frac{1}{2j}.e^{jm(2\pi/N)n} - \frac{1}{2j}.e^{-jm(2\pi/N)n} \qquad \dots (8)$$

using backward Fourier transformation:

$$s(nT) = \frac{1}{N}\sum_{k=0}^{N-1} \dot{S}(k\Omega).e^{j\frac{2\pi}{N}kn}; (n = 0,1,2,..., N-1) \qquad \dots (9)$$

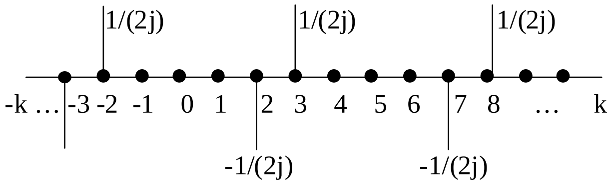
we can determine that:

$$\dot{S}(k\Omega)=\frac{1}{2j}\text{, for } k=m \qquad \text{… (10)}$$

$$\dot{S}(k\Omega)=-\frac{1}{2j}\text{, for } k=-m \qquad \text{… (11)}$$

$$\dot{S}(k\Omega)=0\text{, for } k\neq \pm m \qquad \text{… (12)}$$

For example if *m*=3 and *N*=5



The Metal Detector works in the case like shown above - with exactly recognizable sin waves.

From the other hand we can present the sin(x) by Maclaurin's series:

$$f(x)=\sum_{l=0}^{\infty}\frac{f^{(l)}(0)}{l!}.x^{l} \qquad \text{… (13)}$$

$$f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^{2}+...+\frac{f^{(l)}(0)}{l!}x^{l}+R_{l}(x) \qquad \text{… (14)}$$

$$R_{l}(x)=\frac{x^{l+1}}{(l+1)!}.f^{(l+1)}(x\theta)\text{, where } 0<\theta<1 \qquad \text{… (15)}$$

$$f(x)=\sin(x) \qquad \text{… (16)}$$

Applying (16) in (13) ((14) and (15)) we get:

$$\sin(x)=\sum_{l=0}^{\infty}(-1)^{l+1}\frac{x^{2l-1}}{(2l-1)} \qquad \text{… (17)}$$

or:

$$\sin(x)=\frac{x}{1!}-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\frac{x^{9}}{9!}+...+R_{l}(x) \qquad \text{… (18)}$$

Now we can say that:

$$\sin(x)=A(x)+R_{l}(x), \qquad \text{… (19)}$$

where:

$$A(x)=\frac{x}{1!}-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}+\frac{x^{9}}{9!}+... \qquad \text{… (20)}$$

The sign '…' in (18) and (20) depends on accuracy of the computer and is represented by *l*.

The metal detector works in this way: We send the sin(x) wave with exactly recognizable frequency. We apply FFT over received returned sin(x+**ph**). The **ph** represent the kind of the metal object, which has reflected the wave sent by the detector.

It is clear that when we apply sin(x) to the metal detector (every computer), actually we apply A(x) instead of sin(x). The sign '…' depends on ADC and calculation accuracy. That's why to describe the real signal processing in digital circuits we should not use sin(x), but A(x) in (1) (as Metal Detector reaches (10)-(12) by the way: (1)-(7)).

From the other hand from (19):

$$A(x) = \sin(x) - R_l(x) \qquad \dots (21)$$

Applying (21) in (1), and as in our case $x \equiv t$, therefore $\sin(x) \equiv \sin(t)$ for discrete signal we get:

$$s_d(t) = \sum_{n=0}^{\infty} (\sin(nT) - R_l(nT)).\delta(t - nT) \qquad \dots (22)$$

We apply the discrete Fourier transformation over(22):

$$\dot{S}(\omega) = \int_0^{\infty} s_d(t).e^{-j\omega t}dt = \int_0^{\infty}\sum_{n=0}^{\infty} (\sin(nT) - R_l(nT)).\delta(t - nT).e^{-j\omega t}dt \qquad \dots (23)$$

$$\dot{S}(\omega) = \sum_{n=0}^{\infty} (\sin(nT) - R_l(nT))\int_0^{\infty} \delta(t - nT).e^{-j\omega t}dt \qquad \dots (24)$$

As it is well known that: $\int_0^{\infty} e^{-j\omega t}.\delta(t - nT)dt = e^{-j\omega nT}$,

we get:

$$\dot{S}(\omega) = \sum_{n=0}^{\infty} \sin(nT).e^{-j\omega nT} - \sum_{n=0}^{\infty} R_l(nT).e^{-j\omega nT} \qquad \dots (25)$$

The following frequency is: $\omega = k\Omega = \dfrac{2\pi k}{NT}$, $\qquad \dots (26)$

then:

$$\dot{S}(k\Omega) = \sum_{n=0}^{\infty} \sin(nT).e^{-j\frac{2\pi}{N}kn} - \sum_{n=0}^{\infty} R_l(nT).e^{-j\frac{2\pi}{N}kn}; (k = 0, \pm 1, \pm 2, \dots, \pm N/2) \qquad \dots (27)$$

That is trivial discrete Fourier transformation in forward direction.
The metal detector works with exactly recognizable frequency, i.e. the metal detector sends (and later receives) a wave = $\sin(\Omega_0 n)$, where $\Omega_0 = 2\pi m/N$, where m and N are $2^{integer}$, i.e. m and N are integer. Therefore we can represent the sum: $\sum_{n=0}^{\infty} \sin(nT).e^{-j\frac{2\pi}{N}kn}$,

in (27) by the way: (8) - (12), where $k = 0, \pm 1, \pm 2, \ldots, \pm N/2$:

$$\sum_{n=0}^{\infty} \sin(nT).e^{-j\frac{2\pi}{N}kn} = \begin{cases} \frac{1}{2j}, k = m \\ -\frac{1}{2j}, k = -m \\ 0, k \neq \pm m \end{cases} \quad \ldots (28)$$

Or using (28), (27) become:

$$\dot{S}(k\Omega) = \begin{cases} \frac{1}{2j} - \sum_{n=0}^{\infty} R_l(nT).e^{-jm\frac{2\pi}{N}n}; (k = m) \\ -\frac{1}{2j} - \sum_{n=0}^{\infty} R_l(nT).e^{jm\frac{2\pi}{N}n}; (k = -m) \\ 0 - \sum_{n=0}^{\infty} R_l(nT).e^{-j\frac{2\pi}{N}kn}; (k \neq \pm m) \end{cases} \quad \ldots (29)$$

where $\sum_{n=0}^{\infty} R_l(nT).e^{-j\frac{2\pi}{N}kn}; (k = 0, \pm 1, \pm 2, \ldots, \pm N/2)$, … (30)

is the difference between the real $\dot{S}(k\Omega)$ and the calculated $\dot{S}(k\Omega)$. The difference depends on the exactness of the ADC and calculations.

Now let us look the situation when there are more sources of waves. Unswitched signal is:

$$s(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + \sin(\omega_3 t) + \ldots + \sin(\omega_p t) \quad \ldots (31)$$

Switching (31) and taking in mind (21) and the preceding explanations and conditions:

$$s_d(t) = \sum_{n=0}^{\infty} \begin{pmatrix} \sin(nT_1) + \sin(nT_2) + \sin(nT_3) + \ldots + \sin(nT_p) - R_l(nT_1) - \\ - R_l(nT_2) - R_l(nT_3) - \ldots - R_l(nT_p) \end{pmatrix} .\delta(t - nT) \quad \ldots (32)$$

Applying forward Fourier transformation over (32):

$$\begin{aligned} \dot{S}(\omega) &= \sum_{n=0}^{\infty} \sin(nT_1).e^{-j\omega nT} + \sum_{n=0}^{\infty} \sin(nT_2).e^{-j\omega nT} + \ldots + \sum_{n=0}^{\infty} \sin(nT_p).e^{-j\omega nT} - \\ &- \sum_{n=0}^{\infty} R_l(nT_1).e^{-j\omega nT} - \sum_{n=0}^{\infty} R_l(nT_2).e^{-j\omega nT} - \ldots - \sum_{n=0}^{\infty} R_l(nT_p).e^{-j\omega nT} \end{aligned} \quad \ldots (33)$$

$$\dot{S}(\omega) = \sum_{i=1}^{p} \sum_{n=0}^{\infty} \sin(nT_i).e^{-j\omega nT} - \sum_{i=1}^{p} \sum_{n=0}^{\infty} R_l(nT_i).e^{-j\omega nT} \quad \ldots (34)$$

Including (26)

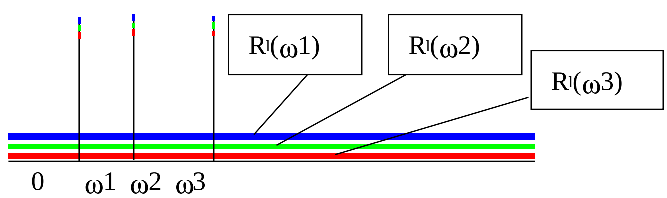
$$\dot{S}(k\Omega)=\sum_{i=1}^{p}\sum_{n=0}^{\infty}\sin(nT_i).e^{-j\frac{2\pi}{N}kn}-\sum_{i=1}^{p}\sum_{n=0}^{\infty}R_l(nT_i).e^{-j\frac{2\pi}{N}kn};(k=0,\pm1,\pm2,...,\pm N/2) \quad \dots (35)$$

Now for simplification all applied frequencies are exactly recognizable, i.e. $T_i = m_i.T$, where m_i and T are integer, and T is sampling period. With other words we had $\sin(\Omega_i n)$, where $\Omega_1 = 2\pi m_1/N$, $\Omega_2 = 2\pi m_2/N$, …, $\Omega_p = 2\pi m_p/N$; $m_{i,(i=1,2,...,p)}$ and N are integer numbers.

We will describe (35) by the way (8)-(12) only for $k \geq 0$, as $k<0$ repeat $k \geq 0$ only with changed sign, and doesn't carry additional information:

$$\dot{S}_d(k\Omega)=\begin{cases}\frac{1}{2j}-\sum_{i=1}^{p}\sum_{n=0}^{\infty}R_l(nT_i).e^{-jm_1\frac{2\pi}{N}n};(k=m_1)\\ \frac{1}{2j}-\sum_{i=1}^{p}\sum_{n=0}^{\infty}R_l(nT_i).e^{-jm_2\frac{2\pi}{N}n};(k=m_2)\\ \dots\\ \frac{1}{2j}-\sum_{i=1}^{p}\sum_{n=0}^{\infty}R_l(nT_i).e^{-jm_p\frac{2\pi}{N}n};(k=m_p)\\ 0-\sum_{i=1}^{p}\sum_{n=0}^{\infty}R_l(nT_i).e^{-j\frac{2\pi}{N}kn};(k\neq m_i,(i=1,..,p))\end{cases} \quad \dots (36)$$

It easy to be seen that when the number of source signal (p) increases, $\dot{S}(k\Omega)$ for $(k \neq m_i,(i=1,..,p))$, increases too instead of being zero, by the formula $\dot{S}(k\Omega)=-\sum_{i=1}^{p}\sum_{n=0}^{\infty}R_l(nT_i).e^{-j\frac{2\pi}{N}kn}$, and $\dot{S}(k\Omega)$ for $(k=m_i,(i=1,..,p))$ decrease by the same way. This means that $p\uparrow \rightarrow wrong\uparrow$ and $p\uparrow \rightarrow recognition\downarrow$.



This boring characteristic of digitizing is very useful for the metal detector. By it we could determine increasing and decreasing of sources of signal with great accuracy in large area of noise tolerance. In the metal detector this method is applied to recognize whether under the metal detector during calibration of the device has had a metal object or now under it has source of inverted for that metal wave.

It is realized in the file WB.m.

Literature

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